Numerical approach to SUSY quantum mechanics and the gauge/gravity duality

Masanori HANADA[1]

Department of Particle Physics and Astrophysics
Weizmann Institute of Science
Rehovot 76100, Israel

Abstract

We demonstrate that Monte-Carlo simulation is a practical tool to study nonperturbative aspects of supersymmetric quantum mechanics. As an example we study D0-brane quantum mechanics in the context of superstring theory. Numerical data nicely reproduce predictions from gravity side, including the coupling constant dependence of the string α' correction. This strongly suggests the duality to hold beyond the supergravity approximation. Although detail of the stringy correction cannot be obtained by state-of-the-art techniques in gravity side, in the matrix quantum mechanics we can obtain concrete values. Therefore the Monte-Carlo simulation combined with the duality provides a powerful tool to study the superstring theory.

[1] Affiliation since October 2010: Department of Physics, University of Washington, Seattle, WA 98195-1560, USA; mhanada@u.washington.edu

1 Introduction

Supersymmetric Yang-Mills (SYM) theories play prominent roles in theoretical particle physics. Especially, maximally supersymmetric theories are of crucial importance for superstring/M theory [1, 2, 3, 4, 5]. Given that most interesting questions can be answered only through nonperturbative study, it is important to construct theoretical frameworks for that. However, it is not a straightforward task because of the notorious difficulties of lattice supersymmetry (SUSY).

Why is lattice SUSY difficult? In lattice gauge theory, gauge symmetry is kept unbroken. Therefore gauge symmetry breaking terms are not generated by radiative corrections. On the other hand, supersymmetry cannot be preserved completely, because the SUSY algebra contains infinitesimal translation, which is broken on lattice by construction. Therefore, even if a given lattice theory converges to a supersymmetric theory at tree level, SUSY breaking operators can be generated radiatively. In order to control the divergence one needs some exact symmetries at discretized level. In 4d $\mathcal{N} = 1$ pure SYM, by keeping the chiral symmetry one can obtain the correct supersymmetric continuum limit [6][2]. In several extended SYM in less than four dimensions, by keeping a part of supersymmetries intact, SUSY breaking operators are forbidden at least to all order in perturbation theory [8]. (A similar statement holds for the Wess-Zumino model, and numerically tested in [9].)

In one dimension (i.e. supersymmetric matrix quantum mechanics), the situation is much easier. Because the theory is UV finite, one does not have to rely on exact symmetries and hence a simple momentum cutoff prescription works [10]. In fact, as demonstrated in [10], the momentum cutoff method is much more powerful than a usual lattice regularization – the convergence to the continuum and restoration of the supersymmetry become much faster, and the Fourier acceleration, which reduces the critical slowing down, can be implemented without any additional cost. As a result it is possible to perform detailed Monte-Carlo study. In this talk we show the result of the Monte-Carlo simulation of maximally supersymmetric matrix quantum mechanics, which describes a system of multiple D0-branes, and compare it with dual string theory. Numerical data nicely reproduce not only predictions from supergravity but also the string α' correction. This strongly suggests the duality to hold beyond the supergravity approximation. In fact, in the gravity side, the stringy correction cannot be evaluated completely because of the lack of the knowledge on the higher derivative correction to the supergravity action. In the matrix quantum mechanics, however, we can obtain concrete values numerically. Therefore the Monte-Carlo simulation provides a new powerful tool to study the superstring theory.

Although we concentrate on a specific model relevant for the gauge/gravity duality, the same method applies to other theories as well. Our message is *Monte-Carlo simulation is a powerful and practical tool to study (SUSY-) quantum mechanics.*

This talk is organized as follows. In § 2 we briefly review the gauge/gravity duality. In § 3 we explain the simulation method. Then in § 4 we provide the simulation results and their interpretations. The materials treated here appeared previously in [11]. In order to avoid the repetition, we put emphasis on an explanation of the duality, so that the physical meaning of the simulation results become clearer.

[2] For recent numerical studies, see [7].

2 The gauge/gravity duality in the D0-brane system

The gauge/gravity duality conjecture [4, 5] claims *type II string theory on black p-brane background and maximally supersymmetric super Yang-Mills theory in $p+1$ dimensions are equivalent.* When $p = 3$ this is known as the AdS_5/CFT_4 correspondence. In this paper we concentrate on $p = 0$ case. The "derivation" of the correspondence is the same as $p = 3$ – both SYM and (weakly-coupled) superstring theory descriptions are valid in a certain parameter region of a coincident Dp-brane system, and hence they should be equivalent. $U(N)$ super Yang-Mills is obtained in the near horizon limit of the N-coincident Dp-brane system, where bulk degrees of freedom decouple. In the same limit the black p-brane solution in type II supergravity reduces to the near extremal solution, and at large-N and strong 't Hooft coupling the supergravity approximation is valid. The near extremal solution is given by [12, 13][5]

$$
\begin{aligned}
ds^2 &= \alpha' \Bigg\{ \frac{U^{(7-p)/2}}{g_{YM}\sqrt{d_p N}} \left[-\left(1 - \frac{U_0^{7-p}}{U^{7-p}}\right) dt^2 + dy_{\|}^2 \right] \\
&\qquad + \frac{g_{YM}\sqrt{d_p N}}{U^{(7-p)/2}\left(1 - \frac{U_0^{7-p}}{U^{7-p}}\right)} dU^2 + g_{YM}\sqrt{d_p N}\, U^{(p-3)/2} d\Omega_{8-p}^2 \Bigg\}, \qquad (1)
\end{aligned}
$$

$$
e^{\phi} = (2\pi)^{2-p} g_{YM}^2 \left(\frac{d_p g_{YM}^2 N}{U^{7-p}} \right)^{\frac{3-p}{4}} \qquad (2)
$$

where U is radial coordinate perpendicular to the black brane, $y_{\|}$ is the coordinate parallel to the brane, Ω_{8-p} represents the spherical coordinate of the transverse directions and a constant d_p is given by $d_p = 2^{7-2p}\pi^{(9-3p)/2}\Gamma((7-p)/2)$. U_0 represents the horizon of the black brane, which is related to the Hawking temperature T as

$$
T = \frac{(7-p)U_0^{(5-p)/2}}{4\pi\sqrt{d_p \lambda}}. \qquad (3)
$$

Note that the gauge coupling constant g_{YM} has dimension of $(\text{mass})^{(3-p)/2}$. As a result, an effective coupling constant describing the black hole thermodynamics is a dimensionless combination $T^{-(3-p)}\lambda$, where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. For this reason, when $p < 3$ "strong coupling" is equivalent to "low temperature". The dilaton ϕ, and hence the string coupling $g_s = e^{\phi}$, depends on the radial coordinate, except for $p = 3$. In order for the supergravity approximation to be valid, the metric should be weakly curved so that α' correction is negligible, and g_s must be small so that the closed string loop correction is small. Roughly speaking, the latter corresponds to the planar limit ($N \to \infty$ with fixed 't Hooft coupling) and the former is achieved at strong 't Hooft coupling (low temperature).

On the contrary to the superstring, SYM is defined nonperturbatively. Therefore, if the gauge/gravity duality is correct, it provides a nonperturbative formulation of superstring theory. As we will see, this viewpoint is important not only conceptually, but also *practically* – once one puts SYM on computer, stringy correction to the supergravity, which is difficult to evaluate from the string theory, can be calculated numerically. We demonstrate it in § 4.

As an example of the correspondence, let us consider the ADM mass of the black brane per unit volume and the energy density of the gauge theory. They are predicted to be the same value, by identifying the Hawking temperature of the black brane to the temperature in the gauge theory. For $p = 0$, the prediction from the supergravity is

$$\frac{E\lambda^{-1/3}}{N^2} = 7.4(T\lambda^{-1/3})^{14/5}, \tag{4}$$

where the 't Hooft coupling $\lambda = g_{YM}^2 N$ has dimension of (mass)^3, T is the temperature of the system and E is the energy density. This relation is expected to hold at $N^{-10/21} \ll T\lambda^{-1/3} \ll 1$, with which the supergravity approximation is valid around the horizon. Note that the black 0-brane has a positive specific heat, in sharp contrast to the Schwarzschild black hole.

When one applies AdS/CFT duality to solve strongly coupled field theory, the Gubser-Klebanov-Polyakov-Witten (GKPW) relation [14], which relates the anomalous dimension of the operators in CFT to mass of the supergravity modes, is very useful. Similar correspondence is proposed for $p \neq 3$ case as well. The basic idea of the GKPW relation is as follows. Let us consider a system of N-coincident D-branes, which gives black p-brane background, and one probe D-brane put far away from others. In the supergravity, the effect of the probe is described as a perturbation around black brane background, specified by the boundary condition $\{h_i\}$. In gauge theory side, it corresponds to $U(N + 1)$ super Yang-Mills theory whose symmetry is broken to $U(N) \times U(1)$. The supergravity modes $\{h_i\}$ couple to operators $\mathcal{O}_i$ in the gauge theory. For $p = 0$, precise dictionary between field theory operators and supergravity modes can be determined by looking at how "Matrix theory currents" couple to bulk gravity modes [15]: one considers the D0-brane action in weakly curved background, which is the Born-Infeld action, expand it around the flat space to linear order in supergravity modes and see how they couple to Matrix theory operators. For example, bulk metric couples to the energy-momentum tensor. This observation leads to $p = 0$ analogue of the GKPW proposal [16]

$$e^{-S_{SG}[h]} = \left\langle \exp\left(\int dt \sum_i h_i(t)\mathcal{O}_i(t)\right)\right\rangle_{YM}, \tag{5}$$

where $S_{SG}[h]$ is the supergravity action in black zero-brane background as a functional of boundary values of the fields $\{h_i\}$. The l.h.s. is evaluated by solving the classical equation of motion by imposing an appropriate boundary condition specified by $\{h_i\}$. In order for this approximation to be valid, distance between two operators must lie in an appropriate range, so that only weakly curved and small g_s region of the bulk contribute to the correlation function. From this one can calculate correlation functions. At zero temperature, operators corresponding to supergravity modes are expected to follow a power law [16]

$$\left\langle \mathcal{O}(t)\mathcal{O}(t')\right\rangle \propto \frac{1}{|t - t'|^{2\nu+1}}, \tag{6}$$

although the theory is not conformal[3]. The region where the supergravity approximation is valid can be found as follows. As the separation between two operators becomes large, the supergravity mode propagates deeper into the bulk, so that it picks up the effect of large g_s at small U. On the

[3] The power law appears due to the generalized conformal symmetry, which is explained below.

other hand, if separation is too small, contribution from large U region becomes important because the radial coordinate is related to the energy scale in the gauge theory. From these considerations, it turns out the supergravity approximation is justified at $\lambda^{-1/3} \ll |t-t'| \ll \lambda^{-1/3}N^{10/21}$ [16]. Note however that this is just a sufficient condition and the result of the approximation may extends beyond this region. As we will see numerically, this is the case indeed.

Another useful quantity which can be studied by using the duality is the supersymmetric Wilson loop [17][18]. At finite temperature, the loop winding on temporal circle (Polyakov loop),

$$W = P\exp\left(\int dt\,(iA(t) + n_i X_i(t))\right), \tag{7}$$

where $\vec{n} = (n_1, \cdots, n_9)$ is an arbitrary unit vector, is an order parameter of the confinement-deconfinement transition [18]. Scalar fields appears because strings pull D-branes. In gravity side, $\langle W\rangle$ is identified to the exponential of the area of the minimal string world-sheet ending on the loop [17],

$$\log\langle W\rangle = 1.89\left(T\lambda^{-1/3}\right)^{-3/5} + \cdots, \tag{8}$$

where $\cdots$ are possible logarithmic and constant corrections.

In AdS/CFT correspondence, CFT operators are put on the 'boundary' of the near horizon region, $U \to \infty$. This procedure seems to be subtle, however, because one has to take the near horizon limit in order to establish the correspondence and 'boundary' is the place where this procedure may fail. It is widely believed that the conformal symmetry saves the situation; if the correspondence holds at small distance, it extends to longer distance as long as the conformal symmetry exists both in gauge theory and in the supergravity. However, the system is not conformal when $p \neq 3$. Then what can protect the correspondence? Actually the geometry (1) is invariant under the *generalized scale transformation*

$$t, y_{\parallel} \quad\to\quad c^{-1}t, c^{-1}y_{\parallel}, \tag{9}$$

$$U \quad\to\quad cU, \tag{10}$$

$$g_{YM}^2 \quad\to\quad c^{3-p}g_{YM}^2, \tag{11}$$

where $c > 0$. There is a counterpart in gauge theory,

$$A_\mu, X_i \quad\to\quad cA_\mu, cX_i, \tag{12}$$

$$t, x \quad\to\quad c^{-1}t, c^{-1}x, \tag{13}$$

$$g_{YM}^2 \quad\to\quad c^{3-p}g_{YM}^2. \tag{14}$$

This generalized conformal symmetry serves as an alternative to the conformal symmetry [19, 20, 16][21]. Note that, for $p \neq 3$, gauge coupling is also scaled reflecting the fact that it is dimensionful. (The effective coupling $\lambda T^{-(3-p)}$ is invariant.) Therefore, in string theory, string coupling constant g_s is rescaled. Hence it is not a "symmetry" in the usual sense; it should be interpreted as a transformation in the whole moduli of string/M theory.

3 Setup for numerical simulation

The action of the maximally supersymmetric matrix quantum mechanics is obtained formally by dimensionally reducing 10d super Yang-Mills theory to 1d:

$$S = \frac{1}{g_{YM}^2} \int_0^\beta dt\, Tr \left\{ \frac{1}{2}(D_t X_i)^2 - \frac{1}{4}[X_i, X_j]^2 + \frac{1}{2}\psi_\alpha D_t \psi_\alpha - \frac{1}{2}\psi_\alpha \gamma_i^{\alpha\beta}[X_i, \psi_\beta] \right\} , \tag{15}$$

where $D_t = \partial_t - i\,[A(t), \ \cdot\ ]$ represents the covariant derivative with the gauge field $A(t)$ being an $N \times N$ Hermitian matrix. It can be viewed as a one-dimensional $U(N)$ gauge theory with adjoint matters. The bosonic matrices $X_i(t)$ $(i = 1, \cdots, 9)$ come from spatial components of the 10d gauge field, while the fermionic matrices $\psi_\alpha(t)$ $(\alpha = 1, \cdots, 16)$ come from a Majorana-Weyl spinor in 10d. The 16×16 matrices γ_i in (15) act on spinor indices and satisfies the Euclidean Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. When we study the energy density and the Wilson loop, we are interested in the finite temperature. Therefore, we impose periodic and anti-periodic boundary conditions on the bosons and fermions, respectively. The extent β in the Euclidean time direction then corresponds to the inverse temperature $\beta \equiv 1/T$. For evaluation of correlators, we impose periodic boundary condition for both bosons and fermions. The coupling constant g_{YM} in (15) can always be scaled out by an appropriate rescaling of the matrices and the time coordinate t. We take $g_{YM} = \frac{1}{\sqrt{N}}$ $(\lambda = 1)$ without loss of generality.

We take the static diagonal gauge $A(t) = \frac{1}{\beta}\mathrm{diag}(\alpha_1, \cdots, \alpha_N)$, where α_a can be chosen to satisfy the constraint $\max_a(\alpha_a) - \min_a(\alpha_a) \le 2\pi$ by using the large gauge transformation with a non-zero winding number. We have to add to the action a term

$$S_{\mathrm{FP}} = -\sum_{a<b} 2 \ln \left| \sin \frac{\alpha_a - \alpha_b}{2} \right| , \tag{16}$$

which appears from the Faddeev-Popov procedure, and the integration measure for α_a is taken to be uniform.

At finite temperature, we make a Fourier expansion

$$X_i^{ab}(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}_{in}^{ab} e^{i\omega n t} \ ; \ \psi_\alpha^{ab}(t) = \sum_{r=-(\Lambda-1/2)}^{\Lambda-1/2} \tilde{\psi}_{\alpha r}^{ab} e^{i\omega r t} . \tag{17}$$

The indices n and r take integer and half-integer values, respectively, corresponding to the imposed boundary conditions.[4] Introducing a shorthand notation

$$\left(f^{(1)} \cdots f^{(p)}\right)_n \equiv \sum_{k_1+\cdots+k_p=n} f_{k_1}^{(1)} \cdots f_{k_p}^{(p)} , \tag{19}$$

we can write the action (15) as $S = S_{\mathrm{b}} + S_{\mathrm{f}}$, where

$$S_{\mathrm{b}} = N\beta \left[\frac{1}{2} \sum_{n=-\Lambda}^{\Lambda} \left(n\omega - \frac{\alpha_a - \alpha_b}{\beta} \right)^2 \tilde{X}_{i,-n}^{ba} \tilde{X}_{in}^{ab} - \frac{1}{4} Tr \left([\tilde{X}_i, \tilde{X}_j]^2 \right)_0 \right],$$

[4] When we impose periodic boundary condition for fermions, they are expanded as

$$\psi_\alpha^{ab}(t) = \sum_{r=-\Lambda}^{\Lambda} \tilde{\psi}_{\alpha n}^{ab} e^{i\omega n t} . \tag{18}$$

$$S_{\rm f} = \frac{1}{2} N\beta \sum_{r=-(\Lambda-1/2)}^{\Lambda-1/2} \left[i \left(r\omega - \frac{\alpha_a - \alpha_b}{\beta} \right) \bar{\tilde{\psi}}^{ba}_{\alpha r} \tilde{\psi}^{ab}_{\alpha r} - (\gamma_i)_{\alpha\beta} Tr \left\{ \bar{\tilde{\psi}}_{\alpha r} \left([\tilde{X}_i, \tilde{\psi}_\beta] \right)_r \right\} \right] . \tag{20}$$

The fermionic action $S_{\rm f}$ may be written in the form $S_{\rm f} = \frac{1}{2}\mathcal{M}_{A\alpha r;B\beta s}\tilde{\psi}^A_{\alpha r}\tilde{\psi}^B_{\beta s}$, where we have expanded $\tilde{\psi}_{\alpha r} = \sum_{A=1}^{N^2} \tilde{\psi}^A_{\alpha r} t^A$ in terms of $U(N)$ generators t^A. Integrating out the fermionic variables, one obtains the Pfaffian Pf$\mathcal{M}$, which is complex in general. However, we observe that it is actually real positive with high accuracy in the temperature regime studied in § 4.2. At very low temperature relevant for § 4.3, and with periodic boundary condition adopted in § 4.4, the phase of the Pfaffian does fluctuate violently. Here we simply neglect the phase and use $|{\rm Pf}\mathcal{M}| = \det(\mathcal{D}^{1/4})$, where $\mathcal{D} = \mathcal{M}^\dagger\mathcal{M}$, instead of Pf$\mathcal{M}$. Surprisingly, it leads to the results expected by the duality[5].

To simulate this system treating fermions fully dynamically, we adopt the RHMC algorithm [22]. The trick of the RHMC is to use the rational approximation $x^{-1/4} \simeq b_0 + \sum_{k=1}^{Q} \frac{a_k}{x+b_k}$, which has sufficiently small relative error within a certain range required by the system to be simulated. (The real positive parameters a_k and b_k can be obtained by a code [23] based on the Remez algorithm.) Then the Pfaffian is replaced by $|{\rm Pf}\mathcal{M}| = \int dF dF^* \exp\left(-S_{\rm PF}\right)$, where

$$S_{\rm PF} = b_0 F^* F + \sum_{k=1}^{Q} a_k F^* (\mathcal{D} + b_k)^{-1} F \ , \tag{21}$$

using the auxiliary complex bosonic variables F, which are called the pseudo-fermions.

We apply the usual HMC algorithm to the whole system as described in [24], except that now we introduce the momentum variables conjugate to the pseudo-fermions F as well as the bosonic matrices $\tilde{X}_i$ and the gauge variables α_a.

When we solve the auxiliary classical Hamiltonian dynamics, low momentum modes tend to evolve larger amount compared to high momentum modes. Therefore, by taking the step sizes of the evolution of lower momentum modes larger, configuration space can be swept out more efficiently. This method is called the Fourier acceleration [25]. In the lattice gauge theory, in order to apply the Fourier acceleration one has to transform the configuration to the momentum representation. On the other hand, in the momentum cutoff method, because we are working directly in the momentum representation, the Fourier acceleration can be implemented without any additional cost. Thanks to this advantage the simulation cost can be reduced drastically.

The main part of the computation comes from solving a linear system $(\mathcal{D} + b_k)\chi = F \quad (k = 1, \cdots, Q)$. We solve the system for the smallest b_k using the conjugate gradient method, which reduces the problem to the iterative multiplications of $\mathcal{M}$ to a pseudo-fermion field, each of which requires $O(\Lambda^2 N^3)$ arithmetic operations if implemented carefully. The solution for larger b_k's can be obtained as by-products using the idea of the multi-mass Krylov solver [26]. This avoids the factor of Q increase of the computational effort.

[5] The agreement suggests the phase quenching is valid, but is not a "proof". To justify it one should look at the correlation between phase and values of physical observables [33].

4 Results

4.1 (In)stability of the black hole

Because the theory is supersymmetric, it has a flat direction along which scalar fields commute each other. As a result, partition function is divergent even at finite volume. Then how can we study this system numerically?

String theory interpretation of the flat direction is obvious. Because scalar eigenvalues corresponds to the positions of D0-branes, it corresponds to a gas of D0-branes. Once they spread widely, there is no force between them because of the supersymmetry and hence each D0-brane propagates freely. Black 0-brane, which is considered in the gauge/gravity duality, is a bound state of many D0-branes. In the matrix model, it is a bound state of scalar eigenvalues. Because the supergravity approximation is valid at large-N, and there the black 0-brane is stable, we expect there is a stable bound state of eigenvalues at large-N. At finite-N, because the string coupling constant is finite, closed string can propagate and sometimes it escapes from the black 0-brane. In other words, scalar eigenvalues can run away from the bound state. Hence the bound state should be at most metastable at finite-N, i.e. black hole is unstable due to the quantum stringy effect.

This is exactly what we observe in numerical simulation. For fixed N, we observe a metastable state at high and low temperature, while at intermediate temperature eigenvalues spreads quickly. To obtain expectation values, we use only configurations in metastable states. As N increases, the metastable state becomes stabler and it exists in wider parametric region.

4.2 Energy vs ADM mass

In order to calculate the energy density efficiently, we use a trick introduced in [24]. In the momentum cutoff prescription, one obtains [27]

$$E = -3T\left\{\langle S_b\rangle - \frac{9}{2}\left((2\Lambda+1)N^2-1\right)\right\}. \tag{22}$$

As we will see shortly, our data is precise enough so that we can evaluate the deviation from the supergravity limit quantitatively. In the dual gravity point of view, the deviation at large-N and at finite 't Hooft coupling corresponds to the string α' correction. Higher derivative corrections to the type IIA supergravity start with α'^3 order [28]. By a simple dimensional counting, it takes the form $(\alpha'/R^2)^3$, where R is the curvature radius of the black brane background, which translates to $(T\lambda^{-1/3})^{9/5}$. Therefore, at large-N, correction to the supergravity limit $E/N^2 = 7.41(T\lambda^{-1/3})^{14/5}$ should be $c(T\lambda^{-1/3})^{23/5}$, where c is an unknown constant.

In Fig. 1 and Fig. 2 we plot the energy density obtained from our simulation [29]. Fig. 1 is a log-log plot of $7.41T^{14/5} - E/N^2$ versus T. At sufficiently low temperature, a power law behavior can be seen clearly (i.e. data points form a straight line), which indicates that higher order corrections are negligible. (A deviation from a power law at very low temperature is merely a finite cutoff effect. Indeed by increasing the momentum cutoff from $\Lambda = 6$ to $\Lambda = 8$ data points go closer to the straight line.) By fitting the result with an ansatz

$$\frac{E}{N^2} = 7.41T^{14/5} - cT^p, \tag{23}$$

we obtain $p = 4.58(3)$ and $c = 5.55(7)$. If we fix $p = 23/5$ instead, we obtain $c = 5.58(1)$. It strongly suggests that the duality is correct including the α' correction. Also, by assuming the duality is correct at this level, Monte-Carlo simulation provides very powerful tool to study the stringy correction to the black hole thermodynamics. Note that the coefficient c cannot be determined from gravity side, because the detail of the higher derivative correction to the type IIA supergravity is not known. Therefore our numerical result can be regarded as a *prediction from gauge theory side*. Usually people use the gauge/gravity duality to calculate something difficult (strongly coupled gauge theory) by dealing with easier problem (supergravity). However here we can use something difficult to study something more difficult (string theory).

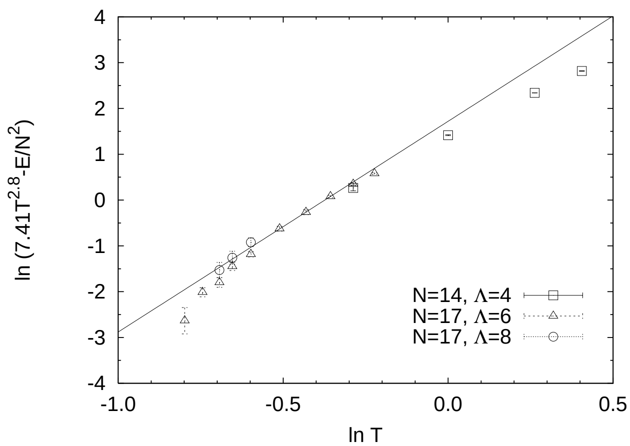

Figure 1: The deviation of the internal energy $\frac{1}{N^2}E$ from the leading term $7.41\,T^{\frac{14}{5}}$ is plotted against the temperature in the log-log scale for $\lambda = 1$. The solid line represents a fit to a straight line with the slope 23/5 predicted from the α' corrections on the gravity side.

4.3 Wilson loop

Next let us consider the Wilson loop [30]. It turned out that agreement with dual gravity prediction emerges only at much lower temperature compared to the case of the energy density. In such a low temperature region, because we have to take the cutoff Λ to be large, we could study only rather small values of N, up to $N = 8$.

In Fig. 3, data points are shown in high and low temperature regions. At intermediate temperature, simulation with modest values of N is unstable because of the scalar instability. Note that we have evaluated $\langle \log |W| \rangle$ instead of $\log\langle W \rangle$ in order to reduce numerical error. We have taken the absolute value because at finite-N arbitrary phase factor can emerge due to the center symmetry and hence $\langle W \rangle = 0$. At large-N, tunneling to different phase is suppressed and $\langle W \rangle$ agrees with $\langle |W| \rangle$ up to a fixed phase factor. That we have taken a logarithm before taking the expectation value can be justified at large-N, where fluctuation is suppressed.

As can be seen from Fig. 3, logarithm of the Wilson loop behaves as $1.89T^{-3/5} + const.$ In fact it is difficult to distinguish constant and $\log T$ from our data; from stringy correction and quantum fluctuation of the string world-sheet, both constant and logarithmic corrections should

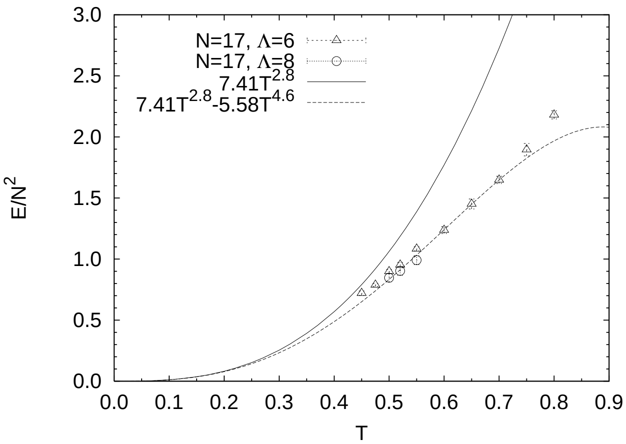

Figure 2: The internal energy $\frac{1}{N^2}E$ is plotted against T for $\lambda = 1$. The solid line represents the leading asymptotic behavior at small T predicted by the gauge-gravity duality. The dashed line represents a fit to the behavior (23) including the sub-leading term with $p = 23/5$ (fixed) and $C = 5.58$.

arise. We consider the "constant" is actually sum of these corrections.

4.4 Correlators

Next we show simulation result for two-point functions [31]. Here we consider a class of operators J^{+ij} given by

$$J^{+ij}_{l,i_1\cdots i_l} \equiv \frac{1}{N} Str\left(F_{ij}X_{i_1}\cdots X_{i_l}\right) \qquad (l \geq 1), \tag{24}$$

which couples to NS-NS 2-form and R-R 1-form. The exponent ν predicted from the supergravity is [16]

$$\nu = \frac{2l}{5}. \tag{25}$$

In the simulation, first we obtain the correlation function in the momentum space,

$$\left\langle \tilde{J}^+_l(p)\tilde{J}^+_l(-p)\right\rangle \qquad (p = 2\pi n/\beta;\ n = 0, 1, 2, \cdots). \tag{26}$$

Because of the existence of the momentum cutoff Λ, the result can be trusted only at small p. In order to estimate the correlators at large p, we assume the behavior

$$\left\langle \tilde{J}^+_l(p)\tilde{J}^+_l(-p)\right\rangle = \frac{a}{p^2}, \tag{27}$$

where a is a constant, and we determine coefficients a by fitting a few points close to Λ.

In order to obtain the correlators in the coordinate space, we perform the Fourier transformation,

$$\langle J^+_l(t)J^+_l(0)\rangle = \left\langle \tilde{J}^+_l(0)\tilde{J}^+_l(0)\right\rangle + \sum_{p>0} 2\cos(pt)\left\langle \tilde{J}^+_l(p)\tilde{J}^+_l(-p)\right\rangle. \tag{28}$$

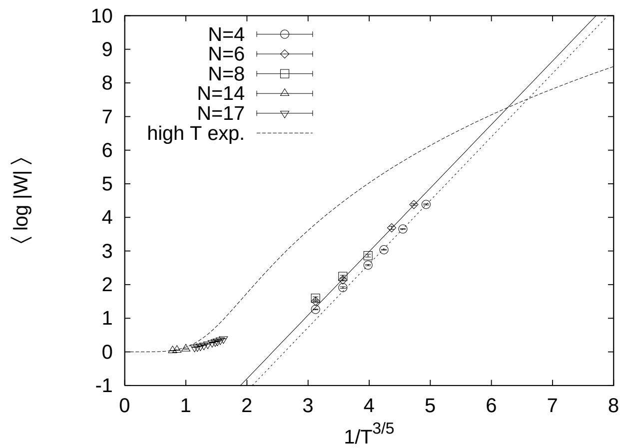

Figure 3: The plot of $\langle \log |W| \rangle$ for $\lambda = 1$ against $T^{-3/5}$. The cutoff Λ is chosen as follows: $\Lambda = 12$ for $N = 4$; $\Lambda = 0.6/T$ for $N = 6, 8$; $\Lambda = 4$ for $N = 14$; $\Lambda = 6$ for $N = 17$. The dashed line represents the results of the high-temperature expansion up to the next-leading order with extrapolations to $N = \infty$, which are obtained by applying the method in [32]. The solid line and the dotted line represent fits for $N = 6$ and $N = 4$ respectively, to straight lines with the slope 1.89 predicted from the gravity side at the leading order.

We terminated the sum w.r.t. p at $p = \frac{2\pi}{\beta} \times 1000$.

As a concrete example, we consider correlators at $N = 3, \Lambda = 16, \beta = 4$. For $n > 12$, we obtain the two-point function by extrapolating the data with the ansatz (27). For the fitting, $n = 10, 11$ and 12 are used. We evaluated the error by using the Jack-knife method, by dividing samples to four bins. The result after the Fourier transformation is shown in Fig. 4. Straight lines represents a power law behavior proportional to $1/t^{2\nu+1}$, where $\nu = 2l/5$ is the prediction from the supergravity. Surprisingly, the expected power appears at such a small value of N. In fact the power law is reproduced with remarkable precision, already at $N = 2$ [33]. Furthermore, the power law extends beyond the parameter region discussed in § 2. It is very interesting because it suggests the gravity prediction can shed a light on deep IR region, which is relevant for the matrix theory conjecture for M-theory [1].

5 Conclusion and discussions

In this talk we explained recent Monte Carlo simulations for maximally supersymmetric matrix quantum mechanics, in the context of the superstring theory[6]. The data reproduces predictions by the gauge/gravity duality precisely, and furthermore it provides a nontrivial prediction for stringy correction to black hole thermodynamics.

There are many future directions. First of all, the same simulation techniques can be applied to other quantum mechanical system as well. It would be nice if one can learn nonperturbative aspects of interesting theories which cannot be calculated analytically. It is also interesting to study

[6] The same system has been studied in [34] and qualitatively consistent results have been obtained.

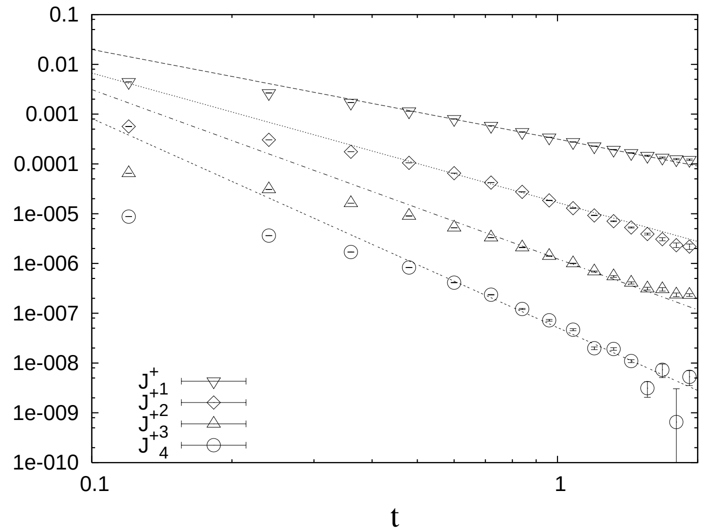

Figure 4: Log-log plot of the correlator $\langle J_i^+(t) J_i^+(0)\rangle$ $(i = 1, 2, 3, 4)$ for $N = 3, \Lambda = 16, \beta = 4$.

theories in higher than one dimension. For example three-dimensional maximally supersymmetric Yang-Mills theory is realized [35] around fuzzy sphere background of the plane wave matrix model [36], and hence can be studied with the momentum cutoff method[7]. Another background in the same matrix model is argued to correspond to 4d $\mathcal{N} = 4$ SYM in the planar limit [38, 11]. In two dimensions, lattice formulation works without fine tuning [37, 39, 40] to all order in perturbation theory, and numerical studies so far confirms it works also at nonperturbative level. (For 4-SUSY system, absence of fine tuning is confirmed in [41], by using dynamical fermion, for two independent lattice formulations. The conservation of the supercurrent has also been confirmed [42].) From string theory point of view, 2d SYM is related [43, 44] to the black hole/black string transition [45]. Recent numerical study in this context can be found in [41, 46]. Finally, 4d $\mathcal{N} = 4$ SYM at finite-N level is realized [47] by combining a matrix model technique [35] and two-dimensional lattice[8]. Together with [38], it provides nonperturbative tools to study AdS_5/CFT_4 correspondence. Monte-Carlo study of these models is of crucial importance[9].

Acknowledgments

I would like to thank all the collaborators involved in the projects reported here. In particular, I am grateful to Jun Nishimura for useful comments on this proceedings. I would also like to thank the organizers of the Workshop “Supersymmetric Quantum Mechanics and Spectral Design” for warm hospitality.

[7] A lattice formulation can be found in [37].

[8] In usual lattice formulation, there is no obvious reason for the absence of the fine tuning. However explicit calculation shows it is absent at one-loop level, and an attempt to provide all-loop argument is going on [48].

[9] For another approach to AdS_5/CFT_4 using simplified model see section V of [49] and references therein.

References

[1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A conjecture," Phys. Rev. D **55**, 5112 (1997), [arXiv:hep-th/9610043].

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A large-N reduced model as superstring," Nucl. Phys. B **498**, 467 (1997), [arXiv:hep-th/9612115].

[3] L. Motl, "Proposals on nonperturbative superstring interactions," arXiv:hep-th/9701025; R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, "Matrix string theory," Nucl. Phys. B **500**, 43 (1997), [arXiv:hep-th/9703030].

[4] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231 (1998); [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[5] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, Phys. Rev. D **58**, 046004 (1998).

[6] D. B. Kaplan, "Dynamical Generation Of Supersymmetry," Phys. Lett. **B136**, 162 (1984).

[7] J. Giedt, R. Brower, S. Catterall, G. T. Fleming and P. Vranas, "Lattice super-Yang-Mills using domain wall fermions in the chiral limit," Phys. Rev. D **79**, 025015 (2009) [arXiv:0810.5746 [hep-lat]].

M. G. Endres, "Dynamical simulation of N=1 supersymmetric Yang-Mills theory with domain wall fermions," Phys. Rev. D **79**, 094503 (2009) [arXiv:0902.4267 [hep-lat]].

K. Demmouche, F. Farchioni, A. Ferling, I. Montvay, G. Munster, E. E. Scholz and J. Wuilloud, "Simulation of 4d N=1 supersymmetric Yang-Mills theory with Symanzik improved gauge action and stout smearing," arXiv:1003.2073 [hep-lat].

[8] D. B. Kaplan, E. Katz and M. Unsal, "Supersymmetry on a spatial lattice," JHEP **0305**, 037 (2003).

[9] S. Catterall and S. Karamov, "Exact lattice supersymmetry: the two-dimensional N = 2 Wess-Zumino model," Phys. Rev. D **65**, 094501 (2002) [arXiv:hep-lat/0108024].

[10] M. Hanada, J. Nishimura and S. Takeuchi, "Non-lattice simulation for supersymmetric gauge theories in one dimension," Phys. Rev. Lett. **99**, 161602 (2007), arXiv:0706.1647 [hep-lat].

[11] J. Nishimura, "Non-lattice simulation of supersymmetric gauge theories as a probe to quantum black holes and strings," PoS **LAT2009**, 016 (2009) [arXiv:0912.0327 [hep-lat]].

[12] G. W. Gibbons and K. i. Maeda, "Black Holes And Membranes In Higher Dimensional Theories With Dilaton Fields," Nucl. Phys. B **298**, 741 (1988).

[13] G. T. Horowitz and A. Strominger, "Black strings and P-branes," Nucl. Phys. B **360**, 197 (1991).

[14] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[15] D. N. Kabat and W. Taylor, Phys. Lett. B **426**, 297 (1998); W. Taylor and M. Van Raamsdonk, JHEP **9904**, 013 (1999); Nucl. Phys. B **558**, 63 (1999).

[16] Y. Sekino and T. Yoneya, Nucl. Phys. B **570**, 174 (2000).
Y. Sekino, Nucl. Phys. B **602**, 147 (2001)

[17] S. J. Rey and J. T. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," Eur. Phys. J. C **22**, 379 (2001) [arXiv:hep-th/9803001].
J. M. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80**, 4859 (1998) [arXiv:hep-th/9803002].

[18] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[19] A. Jevicki, Y. Kazama and T. Yoneya, "Quantum metamorphosis of conformal transformation in D3-brane Yang-Mills theory," Phys. Rev. Lett. **81**, 5072 (1998) [arXiv:hep-th/9808039].

[20] A. Jevicki and T. Yoneya, Nucl. Phys. B **535**, 335 (1998).

[21] T. Azeyanagi, M. Hanada, H. Kawai and Y. Matsuo, "Worldsheet Analysis of Gauge/Gravity Dualities," Nucl. Phys. B **816**, 278 (2009) [arXiv:0812.1453 [hep-th]].

[22] M. A. Clark and A. D. Kennedy, "The RHMC algorithm for 2 flavors of dynamical staggered fermions," Nucl. Phys. Proc. Suppl. **129**, 850 (2004) [arXiv:hep-lat/0309084].

[23] M.A. Clark and A.D. Kennedy,
`http://www.ph.ed.ac.uk/~mike/remez,2005`.

[24] S. Catterall and T. Wiseman, "Towards lattice simulation of the gauge theory duals to black holes and hot strings," JHEP **0712**, 104 (2007) [arXiv:0706.3518 [hep-lat]].

[25] S. Catterall and S. Karamov, "Testing a Fourier accelerated hybrid Monte Carlo algorithm," Phys. Lett. B **528**, 301 (2002).

[26] B. Jegerlehner, "Krylov space solvers for shifted linear systems," arXiv:hep-lat/9612014.

[27] K. N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, "Monte Carlo studies of supersymmetric matrix quantum mechanics with sixteen supercharges at finite temperature," Phys. Rev. Lett. **100**, 021601 (2008) [arXiv:0707.4454 [hep-th]].

[28] D.J. Gross and E. Witten, *"Superstring Modifications of Einstein's Equations"*, Nucl. Phys. B **277**, 1 (1986).

[29] M. Hanada, Y. Hyakutake, J. Nishimura and S. Takeuchi, "Higher derivative corrections to black hole thermodynamics from supersymmetric matrix quantum mechanics," Phys. Rev. Lett. **102**, 191602 (2009) [arXiv:0811.3102 [hep-th]].

[30] M. Hanada, A. Miwa, J. Nishimura and S. Takeuchi, "Schwarzschild radius from Monte Carlo calculation of the Wilson loop in supersymmetric matrix quantum mechanics," Phys. Rev. Lett. **102**, 181602 (2009) [arXiv:0811.2081 [hep-th]].

[31] M. Hanada, J. Nishimura, Y. Sekino and T. Yoneya, "Monte Carlo studies of Matrix theory correlation functions," Phys. Rev. Lett. **104**, 151601 (2010) [arXiv:0911.1623 [hep-th]].

[32] N. Kawahara, J. Nishimura and S. Takeuchi, "High temperature expansion in supersymmetric matrix quantum mechanics," JHEP **0712**, 103 (2007) [arXiv:0710.2188 [hep-th]].

[33] M. Hanada, J. Nishimura, Y. Sekino and T. Yoneya, in preparation.

[34] S. Catterall and T. Wiseman, "Black hole thermodynamics from simulations of lattice Yang-Mills theory," Phys. Rev. D **78**, 041502 (2008) [arXiv:0803.4273 [hep-th]].

S. Catterall and T. Wiseman, "Extracting black hole physics from the lattice," arXiv:0909.4947 [hep-th].

[35] J. M. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdonk, "Transverse fivebranes in matrix theory," JHEP **0301**, 038 (2003).

[36] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[37] D. B. Kaplan and M. Unsal, "A Euclidean lattice construction of supersymmetric Yang-Mills theories with sixteen supercharges," JHEP **0509**, 042 (2005) [arXiv:hep-lat/0503039].

[38] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, "N=4 Super Yang-Mills from the Plane Wave Matrix Model," Phys. Rev. D **78**, 106001 (2008). [arXiv:0807.2352 [hep-th]].

G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, "Deconfinement phase transition in N=4 super Yang-Mills theory on $R \times S^3$ from supersymmetric matrix quantum mechanics," Phys. Rev. Lett. **102**, 111601 (2009) [arXiv:0810.2884 [hep-th]].

G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, "Testing a novel large-N reduction for N=4 super Yang-Mills theory on $R \times S^3$," JHEP **0909**, 029 (2009) [arXiv:0907.1488 [hep-th]].

[39] F. Sugino, "Various super Yang-Mills theories with exact supersymmetry on the lattice," JHEP **0501**, 016 (2005) [arXiv:hep-lat/0410035].

S. Catterall, "Lattice formulation of N = 4 super Yang-Mills theory," JHEP **0506**, 027 (2005) [arXiv:hep-lat/0503036].

A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, "Exact extended supersymmetry on a lattice: Twisted N = 2 super Yang-Mills in two dimensions," Phys. Lett. B **633**, 645 (2006) [arXiv:hep-lat/0507029].

[40] H. Suzuki and Y. Taniguchi, "Two-dimensional N = (2,2) super Yang-Mills theory on the lattice via dimensional reduction," JHEP **0510**, 082 (2005) [arXiv:hep-lat/0507019].

[41] M. Hanada and I. Kanamori, "Lattice study of two-dimensional $\mathcal{N} = (2,2)$ super Yang-Mills at large-N," Phys. Rev. D **80**, 065014 (2009). [arXiv:0907.4966 [hep-lat]].

M. Hanada and I. Kanamori, "Absence of sign problem in two-dimensional N=(2,2) super Yang-Mills on lattice," arXiv:1010.2948 [hep-lat].

[42] I. Kanamori and H. Suzuki, "Restoration of supersymmetry on the lattice: Two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory," Nucl. Phys. B **811**, 420 (2009) [arXiv:0809.2856 [hep-lat]].

[43] L. Susskind, "Matrix theory black holes and the Gross Witten transition," arXiv:hep-th/9805115.

[44] O. Aharony, J. Marsano, S. Minwalla and T. Wiseman, "Black hole-black string phase transitions in thermal 1+1-dimensional supersymmetric Yang-Mills theory on a circle," Class. Quant. Grav. **21**, 5169 (2004) [arXiv:hep-th/0406210].

[45] R. Gregory and R. Laflamme, "Black strings and p-branes are unstable," Phys. Rev. Lett. **70**, 2837 (1993) [arXiv:hep-th/9301052].

[46] S. Catterall, A. Joseph and T. Wiseman, "Thermal phases of D1-branes on a circle from lattice super Yang-Mills," arXiv:1008.4964 [hep-th].

[47] M. Hanada, S. Matsuura and F. Sugino, "Two-dimensional lattice for four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills," arXiv:1004.5513 [hep-lat].

M. Hanada, "A proposal of a fine tuning free formulation of 4d N=4 super Yang-Mills," to appear in JHEP [arXiv:1009.0901 [hep-lat]].

[48] S. Catterall, "Twisted lattice supersymmetry and applications to AdS/CFT," arXiv:1010.6224 [hep-lat].

M. Unsal, private communication.

[49] D. Berenstein, arXiv:1010.3270 [hep-th].